

Conflict-Free Coloring Made Stronger

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Abstract. In FOCS 2002, Even et al. showed that any set of n discs in the plane can be Conflict-Free colored with a total of at most $O(\log n)$ colors. That is, it can be colored with $O(\log n)$ colors such that for any (covered) point p there is some disc whose color is distinct from all other colors of discs containing p . They also showed that this bound is asymptotically tight. In this paper we prove the following stronger results:

- (i) Any set of n discs in the plane can be colored with a total of at most $O(k \log n)$ colors such that (a) for any point p that is covered by at least k discs, there are at least k distinct discs each of which is colored by a color distinct from all other discs containing p and (b) for any point p covered by at most k discs, all discs covering p are colored distinctively. We call such a coloring a *k-Strong Conflict-Free* coloring. We extend this result to pseudo-discs and arbitrary regions with linear union-complexity.
- (ii) More generally, for families of n simple closed Jordan regions with union-complexity bounded by $O(n^{1+\alpha})$, we prove that there exists a *k-Strong Conflict-Free* coloring with at most $O(kn^\alpha)$ colors.
- (iii) We prove that any set of n axis-parallel rectangles can be *k-Strong Conflict-Free* colored with at most $O(k \log^2 n)$ colors.
- (iv) We provide a general framework for *k-Strong Conflict-Free* coloring arbitrary hypergraphs. This framework relates the notion of *k-Strong Conflict-Free* coloring and the recently studied notion of *k-colorful* coloring.

All of our proofs are constructive. That is, there exist polynomial time algorithms for computing such colorings.

Key Words. Conflict-Free Colorings, Geometric hypergraphs, Wireless networks, Discrete geometry.

1 Introduction and Preliminaries

Motivated by modeling frequency assignment to cellular antennae, Even et al. [17] introduced the notion of Conflict-Free colorings. A *Conflict-Free* coloring (CF in short) of a hypergraph $H = (V, \mathcal{E})$ is a coloring of the vertices V such that for any non-empty hyperedge $e \in \mathcal{E}$ there is some vertex $v \in e$ whose color is distinct from all other colors of vertices in e . For a hypergraph H , one seeks the least number of colors l such that there exists an l -coloring of H which is Conflict-Free. It is easily seen that CF-coloring of a hypergraph H coincides with the notion of classical graph coloring in the case when H is a graph (i.e., all hyperedges are of cardinality two). Thus it can be viewed as a generalization of graph coloring. There are two well known generalizations of graph coloring to hypergraph coloring in the literature (see, e.g., [9]). The first generalization requires “less” than the CF requirement and this is the *non-monochromatic* requirement where each hyperedge in \mathcal{E} of cardinality at least two should be non-monochromatic: The *chromatic number* of a hypergraph H , denoted $\chi(H)$, is the least number l such that H admits an l -coloring which is a non-monochromatic coloring. The second generalization requires “more” than the CF requirement and this is the *colorful* requirement where each hyperedge should be colorful (i.e., all of its vertices should have distinct colors). For instance, consider the following hypergraph

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$H = (V, \mathcal{E})$: Let $V = \{1, 2, \dots, n\}$ and let \mathcal{E} consist of all subsets of V consisting of consecutive numbers of V . That is, \mathcal{E} consists of all discrete intervals of V . It is easily seen that one can color the elements of V with two colors in order to obtain a non-monochromatic coloring of H . Color the elements of V alternately with ‘black’ and ‘white’. On the other extreme, one needs n colors in any colorful coloring of H . Indeed V itself is also a hyperedge in this hypergraph (an ‘interval’ containing all elements of V) so all colors must be distinct. However, it is an easy exercise to see that there exists a CF-coloring of H with $\lfloor \log n \rfloor + 1$ colors. In fact, for an integer $k > 0$, if V consist of $2^k - 1$ elements then k colors suffice and are necessary for CF-coloring H .

Let \mathcal{R} be a finite collection of regions in \mathbb{R}^d , $d \geq 1$. For a point $p \in \mathbb{R}^d$, define $r(p) = \{R \in \mathcal{R} : p \in R\}$. The hypergraph $(\mathcal{R}, \{r(p)\}_{p \in \mathbb{R}^d})$, denoted $H(\mathcal{R})$, is called the hypergraph *induced* by \mathcal{R} . Such hypergraphs are referred to as *geometrically induced* hypergraphs. Informally these are the Venn diagrams of the underlying regions.

In general, dealing with CF coloring for arbitrary hypergraphs is not easier than graph coloring. The paper [17] focused on hypergraphs that are induced by geometric objects such as discs, squares etc. Their motivation was a modeling of frequency assignment to cellular antennae in a manner that reduces the spectrum of frequencies used by a network of antennae. Suppose that antennae are represented by discs in the plane and that every client (holding a cell-phone) is represented by a point. Antennae are assigned frequencies (this is the coloring). A client is served provided that there is at least one antenna ‘covering’ the client for which the assigned frequency is “unique” and therefore has no “conflict” (interference) with other frequencies used by nearby antennae. When \mathcal{R} is a finite family of n discs in the plane \mathbb{R}^2 , Even et al. [17] proved that finding an optimal CF-coloring for \mathcal{R} is NP-hard. However, they showed that there is always a CF-coloring of $H(\mathcal{R})$ with $O(\log n)$ colors and that this bound is asymptotically tight. That is, for every n there is a family of n discs which requires $\Omega(\log n)$ colors in any CF-coloring. See [17] for further discussion of this model and the motivation.

CF-coloring finds application also in activation protocols for RFID networks. Radio frequency identification (RFID) is a technology where a reader device can sense the presence of a nearby object by reading a tag device attached to the object. To improve coverage, multiple RFID readers can be deployed in the given region. However, two readers trying to access a tagged device simultaneously might cause mutual interference. One may want to design scheduled access of RFID tags in a multiple reader environment. Assume that we have t time slots and we would like to ‘color’ each reader with a time slot in $\{1, \dots, t\}$ such that the reader will try to read all nearby tags in its given time slot. In particular, we would like to read all the tags and minimize the total time slots t . It is easily seen that if we CF-color the family \mathcal{R} of readers then in this coloring every possible tag will have a time slot and a single reader trying to access it in that time slot [18]. The notion of CF-coloring has caught much scientific attention in recent years both from the algorithmic and combinatorial point of view [3,4,6,7,8,11,12,13,16,19,20,?,22,25].

Our Contribution: In this paper we study the notion of *k-Strong-Conflict-Free* (abbreviated, *kSCF*) colorings of hypergraphs. This notion extends the notion of *CF*-colorings of hypergraphs. Informally, in the case of coloring discs, rather than having at least one unique color at every covered point p , we require at least k distinct colors to some k discs such that each of these colors is unique among the discs covering p . The motivation for studying *kSCF*-coloring is rather straightforward in the context of wireless antennae. Having, say $k > 1$ unique frequencies in any given location allows us to serve k clients at that location rather than only one client. In the context of RFID networks, a *kSCF* coloring will correspond to an activation protocol which is fault-tolerant. That is, every tag can be read even if some $k - 1$ readers are broken.

Definition 1 (*k-Strong Conflict-Free coloring*): Let $H = (V, \mathcal{E})$ be a hypergraph and let $k \in \mathbb{N}$ be some fixed integer. A coloring of V is called *k-Strong-Conflict-Free* for H if

- (i) for every hyperedge $e \in \mathcal{E}$ with $|e| \geq k$ there exists at least k vertices in e , whose colors are unique among the colors assigned to the vertices of e , and
- (ii) for each hyperedge $e \in \mathcal{E}$ with $|e| < k$ all vertices in e get distinct colors.

Let $f_H(k)$ denote the least integer l such that H admits a *kSCF*-coloring with l colors.

Note that a CF -coloring of a hypergraph H is $kSCF$ -coloring of H for $k = 1$.

Abellanas et al. [2] were the first to study $kSCF$ -coloring¹. They focused on the special case where V is a finite set of points in the plane and \mathcal{E} consist of all subsets of V which can be realized as an intersection of V with a disc. They showed that in this case the hypergraph admits a $kSCF$ -coloring with $O(\frac{\log n}{\log \frac{ck}{c-1}})$ ($= O(k \log n)$) colors, for some absolute constant c . See also [1].

The following notion of k -colorful colorings was recently introduced and studied by Aloupis et al. [5] for the special case of hypergraphs induced by discs.

Definition 2. Let $H = (V, \mathcal{E})$ be a hypergraph, and let φ be a coloring of H . A hyperedge $e \in \mathcal{E}$ is said to be k -colorful with respect to φ if there exist k vertices in e that are colored distinctively under φ . The coloring φ is called k -colorful if every hyperedge $e \in \mathcal{E}$ is $\min\{|e|, k\}$ -colorful. Let $c_H(k)$ denote the least integer l such that H admits a k -colorful coloring with l colors.

Aloupis et al. were motivated by a problem related to battery lifetime in sensor networks. See [5,10,23] for additional details on the motivation and related problems.

Remark: Every $kSCF$ -coloring of a hypergraph H is a k -colorful coloring of H . However, the opposite claim is not necessarily true. A k -colorful coloring assures us that every hyperedge of cardinality at least k has at least k distinct colors present in it. However, these k colors are not necessarily unique since each may appear with multiplicity.

A k -colorful coloring can be viewed as a type of coloring which is “in between” non-monochromatic coloring and colorful coloring. A 2-colorful coloring of H is exactly the classical non-monochromatic coloring, so $\chi(H) = c_H(2)$. If H is a hypergraph with n vertices, then an n -colorful coloring of H is the classical colorful coloring of H . Consider the hypergraph H , consisting of all discrete intervals on $V = \{1, \dots, n\}$ mentioned earlier. It is easily seen that for any i , an i -colorful coloring with i colors is obtained by coloring V in increasing order with $1, 2, \dots, i, 1, 2, \dots, i, 1, 2, \dots$ with repetition.

In this paper, we study a connection between k -colorful coloring and Strong-Conflict-Free coloring of hypergraphs. We show that if a hypergraph H admits a k -colorful coloring with a “small” number of colors (hereditarily) then it also admits a $(k-1)SCF$ -coloring with a “small” number of colors. The interrelation between the quoted terms is provided in Theorems 1 and 2 below.

Let $H = (V, \mathcal{E})$ be a hypergraph and let $V' \subset V$. We write $H[V']$ to denote the sub-hypergraph of H induced by V' , i.e., $H[V'] = (V', \mathcal{E}')$ and $\mathcal{E}' = \{e \cap V' | e \in \mathcal{E}\}$. We write $n(H)$ to denote the number of vertices of H .

Theorem 1. Let $H = (V, \mathcal{E})$ be a hypergraph with n vertices, and let $k, \ell \in \mathbb{N}$ be fixed integers, $k \geq 2$. If every induced sub-hypergraph $H' \subseteq H$ satisfies $c_{H'}(k) \leq \ell$, then $f_H(k-1) \leq \log_{1+\frac{1}{\ell-1}} n = O(\ell \log n)$.

Theorem 2. Let $H = (V, \mathcal{E})$ be a hypergraph with n vertices, let $k \geq 2$ be a fixed integer, and let $0 < \alpha \leq 1$ be a fixed real. If every induced sub-hypergraph $H' \subseteq H$ satisfies $c_{H'}(k) = O(kn(H')^\alpha)$, then $f_H(k-1) = O(kn(H')^\alpha)$.

Consider the hypergraph of “discrete intervals” with n vertices. As mentioned earlier, it has a $(k+1)$ -colorful coloring with $k+1$ colors and this holds for every induced sub-hypergraph. Thus, Theorem 1 implies that it also admits a $kSCF$ -coloring with at most $\log_{1+\frac{1}{k}} n = O(k \log n)$ colors. In Section 3.1, we provide an upper bound on the number of colors required by $kSCF$ -coloring of geometrically induced hypergraphs as a function of the union-complexity of the regions that induce the hypergraphs. Below we describe the relations between the union-complexity of the regions, k -colorful and $(k-1)SCF$ coloring of the underlying hypergraph. First, we need to define the notion of union-complexity.

¹ They referred to such a coloring as k -Conflict-Free coloring.

Definition 3. For a family \mathcal{R} of n simple closed Jordan regions in the plane, let $\partial\mathcal{R}$ denote the boundary of the union of the regions in \mathcal{R} . The union-complexity of \mathcal{R} is the number of intersection points, of a pair of boundaries of regions in \mathcal{R} , that belong to $\partial\mathcal{R}$.

For a set \mathcal{R} of n simple closed planar Jordan regions, let $\mathcal{U}_{\mathcal{R}} : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $\mathcal{U}_{\mathcal{R}}(m)$ is the maximum union-complexity of any subset of k regions in \mathcal{R} over all $k \leq m$, for $1 \leq m \leq n$. We abuse the definition slightly and assume that the union-complexity of any set of n regions is at least n . When dealing with geometrically induced hypergraphs, we consider k -colorful coloring and $kSCF$ -coloring of hypergraphs that are induced by simple closed Jordan regions having union-complexity at most $O(n^{1+\alpha})$, for some fixed parameter $0 \leq \alpha \leq 1$. The value $\alpha = 0$ corresponds to regions with linear union-complexity such as discs or pseudo-discs (see, e.g., [21]). The value $\alpha = 1$ corresponds to regions with quadratic union-complexity. See [14,15] for additional families with sub-quadratic union-complexity.

In the following theorem we provide an upper bound on the number of colors required by a k -colorful coloring of a geometrically induced hypergraph as a function of k and of the union-complexity of the underlying regions inducing the hypergraph :

Theorem 3. Let $k \geq 2$, let $0 \leq \alpha \leq 1$, and let c be a fixed constant. Let \mathcal{R} be a set of n simple closed Jordan regions such that $\mathcal{U}_{\mathcal{R}}(m) \leq cm^{1+\alpha}$, for $1 \leq m \leq n$, and let $H = H(\mathcal{R})$. Then $c_H(k) = O(kn^\alpha)$.

Combining Theorem 1 with Theorem 3 (for $\alpha = 0$) and Theorem 2 with Theorem 3 (for $0 < \alpha < 1$) yields the following result:

Theorem 4. Let $k \geq 2$, let $0 \leq \alpha \leq 1$, and let c be a constant. Let \mathcal{R} be a set of n simple closed Jordan regions such that $\mathcal{U}_{\mathcal{R}}(m) = cm^{1+\alpha}$, for $1 \leq m \leq n$. Let $H = H(\mathcal{R})$. Then:

$$f_H(k-1) = \begin{cases} O(k \log n), & \alpha = 0, \\ O(kn^\alpha), & 0 < \alpha \leq 1. \end{cases}$$

In Section 3.2 we consider $kSCF$ -colorings of hypergraphs induced by axis-parallel rectangles in the plane. It is easy to see that axis-parallel rectangles might have quadratic union-complexity, for example, by considering a grid-like construction of $n/2$ disjoint (horizontally narrow) rectangles and $n/2$ disjoint (vertically narrow) rectangles. For a hypergraph H induced by axis-parallel rectangles, Theorem 4 states that $f_H(k-1) = O(kn)$. This bound is meaningless, since the bound $f_H(k-1) \leq n$ is trivial. Nevertheless, we provide a near-optimal upper bound for this case in the following theorem:

Theorem 5. Let $k \geq 2$. Let \mathcal{R} be a set of n axis-parallel rectangles, and let $H = H(\mathcal{R})$. Then $f_H(k-1) = O(k \log^2 n)$.

In order to obtain Theorem 5 we prove the following theorem:

Theorem 6. Let $H = H(\mathcal{R})$ be the hypergraph induced by a family \mathcal{R} of n axis-parallel rectangles in the plane, and let $k \in \mathbb{N}$ be an integer, $k \geq 2$. For every induced sub-hypergraph $H' \subseteq H$ we have: $c_{H'}(k) \leq k \log n$.

Theorem 5 is therefore an easy corollary of Theorem 6 combined with Theorem 1.

Har-Peled and Smorodinsky [19] proved that any family \mathcal{R} of n axis-parallel rectangles admits a CF-coloring with $O(\log^2 n)$ colors. Their proof uses the probabilistic method. They also provide a randomized algorithm for obtaining CF-coloring with at most $O(\log^2 n)$ colors. Later, Smorodinsky [25] provided a deterministic polynomial-time algorithm that produces a CF-coloring for n axis-parallel rectangles with $O(\log^2 n)$ colors. Theorem 5 thus generalizes the results of [19] and [25].

All of our proofs are constructive. In other words, there exist deterministic polynomial-time algorithms to obtain the required $kSCF$ coloring with the promised bounds. In this paper, we omit the technical details of the underlying algorithms and we do not make an effort to optimize their running time.

The result of Ali-Abam *et al.*[1] implies that the upper bounds provided in Theorem 4 for $\alpha = 0$ and Theorem 5 are optimal. Specifically, they provide matching lower bounds on the number of colors required by any $kSCF$ -coloring of hypergraphs induced by (unit) discs and axis-parallel squares in the plane by a simple analysis of such coloring for the discrete intervals hypergraph mentioned earlier.

Organization. In Section 2 we prove Theorems 1 and 2. In Section 3.1 we prove Theorems 3 and 4. Finally, in Section 3.2 we prove Theorems 5 and 6.

2 A Framework For Strong-Conflict-Free Coloring

In this section, we prove Theorems 1 and 2. To that end we devise a framework for obtaining an upper bound on the number of colors required by a Strong-Conflict-Free coloring of a hypergraph. Specifically, we show that if there exist fixed integers k and l such that an n -vertex hypergraph H admits the hereditary property that every vertex-induced sub-hypergraph H' of H admits a k -colorful coloring with at most l colors, then H admits a $(k - 1)SCF$ -coloring with $O(l \log n)$ colors. For the case when l is replaced with the function $kn(H')^\alpha$ we get a better bound without the $\log n$ factor.

Framework \mathcal{A} :

Input: A hypergraph H satisfying the conditions of Theorems 1 and 2.

Output: A $(k - 1)SCF$ -coloring of H .

- 1: $i \leftarrow 1$ $\{i$ denotes an unused color. $\}$
- 2: **while** $V \neq \emptyset$ **do**
- 3: **Auxiliary Coloring:** Let $\varphi : V \rightarrow [\ell]$ be a k -colorful coloring of $H[V]$ with at most ℓ colors.
- 4: Let V' be a color class of φ of maximum cardinality.
- 5: **Color:** Set $\chi(u) = i$ for every vertex $u \in V'$.
- 6: **Discard:** $V \leftarrow V \setminus V'$.
- 7: **Increment:** $i \leftarrow i + 1$.
- 8: **end while**
- 9: **Return** χ .

Proof of Theorems 1 and 2. We show that the coloring produced by Framework \mathcal{A} is a $(k - 1)SCF$ -coloring of H with a total number of colors as specified in Theorems 1 and 2.

Let χ denote the coloring obtained by the application of framework \mathcal{A} on H . The number of colors used by χ is the number of iterations performed by \mathcal{A} . By the pigeon-hole principle, at least $|V|/\ell$ vertices are removed in each iteration (where V is the set of vertices remained after the last iteration). Therefore, the total number of iterations performed by \mathcal{A} is bounded by $\log_{1+\frac{1}{\ell-1}} n$. Thus, the coloring χ uses at most $\log_{1+\frac{1}{\ell-1}} n$ colors. If in step 3 of the framework, l is replaced with the function $k|V|^\alpha$ (for a fixed parameter $0 < \alpha < 1$), then by the pigeon-hole principle at least $\frac{|V|^{1-\alpha}}{k}$ vertices of H are discarded in step 6 of that iteration. It is easily seen that the number of iterations performed in this case is bounded by $O(kn^\alpha)$ where $n = n(H)$.

Next, we prove that the coloring χ is indeed a $(k - 1)SCF$ -coloring of H . The colors of χ are the indices of iterations of \mathcal{A} . Let $e \in \mathcal{E}$ be a hyperedge of H . If $|e| \leq k$ then it is easily seen that all colors of vertices of e are distinct. Indeed, by the property of the auxiliary coloring φ in step 3 of the framework, every vertex of e is colored distinctively and in each such iteration, at most one vertex from e is colored by χ so χ colors all vertices of e in distinct iterations. Next, assume that $|e| > k$. We prove that e contains at least $k - 1$ vertices that are assigned unique colors in χ . For an integer r , let $\{\alpha_1, \dots, \alpha_r\}$ denote the r largest colors in decreasing order that are assigned to some vertices of e . That is, the color α_1 is the largest color assigned to a vertex of e , the color α_2 is the second largest color and so on. In what follows, we prove a stronger assertion that for every $1 \leq j \leq k - 1$ the color α_j exists and is unique in e . The proof is by induction on j . α_1

exists in e by definition. For the base of the induction we prove that α_1 is unique in e . Suppose that the color α_1 is assigned to at least two vertices $u, v \in e$, and consider iteration α_1 of \mathcal{A} . Let $H' = H[\{x \in V : \chi(x) \geq \alpha_1\}]$, and let φ be the k -colorful coloring obtained for H' in step 3 of iteration α_1 . Put $e' = \{x \in e : \chi(x) \geq \alpha_1\}$. $e' \subset e$ is a hyperedge in H' . Since $u, v \in e'$ then $|e'| \geq 2$. φ is k -colorful for H' so e' contains at least two vertices that are colored distinctively in φ . In iteration α_1 , the vertices of one color class of φ are removed from e' . Since e' contains vertices from two color classes of φ , it follows that after iteration α_1 at least one vertex of e' remains. Thus, at least one vertex of e' is colored in a later iteration than α_1 , a contradiction to the maximality of α_1 . The induction hypothesis is that in χ the colors $\alpha_1, \dots, \alpha_{j-1}$, $1 < j \leq k-1$, all exist and are unique in the hyperedge e . Consider the color α_j . There exists a vertex $u \in e$ such that $\chi(u) = \alpha_j$; for otherwise it follows from the induction hypothesis that $|e| < k-1$ since the colors $\alpha_1, \dots, \alpha_{j-1}$ are all unique in e and $j-1 < k-1$. We prove that the color α_j is unique in e . Assume to the contrary that α_j is not unique at e , and that in χ the color α_j is assigned to at least two vertices $u, v \in e$. Put $H'' = H[\{u \in V : \chi(u) \geq \alpha_j\}]$, and let φ'' be the k -colorful coloring obtained for H'' in step 3 of iteration α_j . Put $e'' = \{u \in e : \chi(u) \geq \alpha_j\}$. e'' is a hyperedge of H'' . By the induction hypothesis and the definition of the colors $\alpha_1, \dots, \alpha_{j-1}$, after iteration α_j a set $U \subset e''$ of exactly $j-1$ vertices of e'' remains. In addition, $u, v \in e''$ and $U \cap \{u, v\} = \emptyset$. Consequently, $|e''| \geq j+1$. Since φ'' is k -colorful then e'' contains vertices from $\min\{k, j+1\}$ color classes of φ'' . $j \leq k-1$ so $\min\{k, j+1\} = j+1$. Since in iteration α_j the vertices of one color class of φ'' are removed from e'' , it follows that after iteration α_j at least j vertices of e'' remain. This is a contradiction to the induction hypothesis. ■

Remark. Given a k -colorful coloring of H , the framework \mathcal{A} obtains a Strong Conflict-Free coloring of H in a constructive manner. As mentioned above, in this paper, computational efficiency is not of main interest. However, it can be seen that for certain families of geometrically induced hypergraphs, framework \mathcal{A} produces an efficient algorithm. In particular, for hypergraphs induced by discs or axis-parallel rectangles, framework \mathcal{A} produces an algorithm with a low degree polynomial running time. Colorful-colorings of such hypergraphs can be computed once the arrangement of the discs is computed together with the depth of every face (see, e.g., [24]). Due to space limitation we omit the technical details involving the description of these algorithms for computing k -colorful coloring for those hypergraphs.

3 k -Strong-Conflict-Free Coloring of Geometrically Induced Hypergraphs

Theorems 1 and 2 assert that in order to attain upper bounds on $f_H(k)$, for a hypergraph H , one may concentrate on attaining an upper bound on $c_H(k+1)$. In this section we concentrate on colorful colorings.

3.1 k -Strong-Conflict-Free Coloring and Union Complexity

In this section, we prove Theorems 3 and 4. Before proceeding with a proof of Theorem 3, we need several related definitions and theorems. A simple finite graph G is called k -degenerate if every vertex-induced sub-graph of G contains a vertex of degree at most k . For a finite set \mathcal{R} of simple closed planar Jordan regions and a fixed integer k , let $G_k(\mathcal{R})$ denote the graph with vertex set \mathcal{R} and two regions $r, s \in \mathcal{R}$ are adjacent in $G_k(\mathcal{R})$ if there exists a point $p \in \mathbb{R}^2$ such that (i) $p \in r \cap s$, and (ii) there exists at most k regions in $\mathcal{R} \setminus \{r, s\}$ that contain p .

Theorem 7. *Let \mathcal{R} be a finite set of simple closed planar Jordan regions, let $H = H(\mathcal{R})$, and let k be a fixed integer. If $G_k(\mathcal{R})$ is l -degenerate then $c_H(k) \leq l+1$.*

Theorem 7 can be proved in a manner similar to that of Aloupis et al. (see [5]) who proved Theorem 7 in the special case when \mathcal{R} is a family of discs. Due to space limitations, we omit a proof of this theorem.

In light of Theorem 7, in order to prove Theorem 3 it is sufficient to prove that for a family of regions satisfying the conditions of Theorem 3 and a fixed integer k , the graph $G_k(\mathcal{R})$ is $O(kn^\alpha)$ -degenerate, where α is as in Theorem 3.

Lemma 1. *Let $k \geq 0$, let $0 \leq \alpha \leq 1$, and let c be a fixed constant. Let \mathcal{R} be a set of n simple closed Jordan regions such that $\mathcal{U}_{\mathcal{R}}(m) \leq cm^{1+\alpha}$, for $1 \leq m \leq n$. Then $G_k(\mathcal{R})$ is $O(kn^\alpha)$ -degenerate.*

Our approach to proving Lemma 1 requires several steps. These steps are described in the following lemmas. We shall provide an upper bound on the average degree of every vertex-induced subgraph of $G_k(\mathcal{R})$ by providing an upper bound on the number of its edges. We need the following lemma:

Lemma 2. ([25]) *Let \mathcal{R} be a set of n simple closed planar Jordan regions and let $\mathcal{U} : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $\mathcal{U}(m)$ is the maximum union-complexity of any $k \leq m$ regions in \mathcal{R} over all $k \leq m$. Then the average degree of $G_0(\mathcal{R})$ is $O(\frac{\mathcal{U}(n)}{n})$.*

For a graph G , we write $E(G)$ to denote the set of edges of G . We use Lemma 2 to obtain the following easy lemma.

Lemma 3. *Let $0 \leq \alpha \leq 1$ and let c be a fixed constant. Let \mathcal{R} be a set of n simple closed Jordan regions such that $\mathcal{U}_{\mathcal{R}}(m) \leq cm^{1+\alpha}$, for $1 \leq m \leq n$. Then there exists a constant d such that $|E(G_0(\mathcal{R}))| \leq \frac{dn^{1+\alpha}}{2}$.*

Proof: By Lemma 2, it follows that there exists a constant d' such that

$$\frac{2|E(G_0(\mathcal{R}))|}{n} = \frac{\sum_{x \in V(G_0(\mathcal{R}))} \deg_{G_0(\mathcal{R})}(x)}{n} \leq \frac{d'cn^{1+\alpha}}{n}.$$

Set $d = d'c$ and the claim follows. \blacksquare

For a set \mathcal{R} of n simple closed planar Jordan regions, define $I(\mathcal{R})$ to denote the graph whose vertex set is \mathcal{R} and two regions $r, s \in \mathcal{R}$ are adjacent if $r \cap s \neq \emptyset$. The graph $I(\mathcal{R})$ is called the *intersection graph* of \mathcal{R} . Note that for any integer k $E(G_k(\mathcal{R})) \subseteq E(I(\mathcal{R}))$. Let $E \subseteq E(I(\mathcal{R}))$ be an arbitrary subset of the edges of $I(\mathcal{R})$. For every edge $e = (a, b) \in E$, pick a point $p_e \in a \cap b$. Note that for distinct edges e and e' in E it is possible that $p_e = p_{e'}$. Put $X_{E, \mathcal{R}} = \{(p_e, r) : e = (a, b) \in E \text{ and } r \in \mathcal{R} \setminus \{a, b\} \text{ contains } p_e\}$. In the following two lemmas we obtain a lower bound on $|X_{E, \mathcal{R}}|$ in terms of $|E|$ and $|\mathcal{R}|$.

Lemma 4. *Let $0 \leq \alpha \leq 1$ and let c and d be the constants of Lemma 3. Let \mathcal{R} be a set of n simple closed Jordan regions such that $\mathcal{U}_{\mathcal{R}}(m) \leq cm^{1+\alpha}$, for $1 \leq m \leq n$. Let $E \subseteq E(I(\mathcal{R}))$. Then $|X_{E, \mathcal{R}}| \geq |E| - \frac{dn^{1+\alpha}}{2}$.*

Proof: Apply induction on the value $|E| - \frac{dn^{1+\alpha}}{2}$. Let $P_E = \{p_e : e \in E\}$. One may assume that $|E| - \frac{dn^{1+\alpha}}{2} \geq 0$ for otherwise the claim follows trivially since $|X_{E, \mathcal{R}}| \geq 0$. Suppose $|E| - \frac{dn^{1+\alpha}}{2} = 1$. Since $|E| > \frac{dn^{1+\alpha}}{2}$, then by Lemma 3 there exists an edge $e = (a, b) \in E \setminus E(G_0(\mathcal{R}))$. Since $e \notin E(G_0(\mathcal{R}))$, it follows that for every point $p \in a \cap b$ there exists a region $r \in \mathcal{R} \setminus \{a, b\}$ such that $p \in r$. Consequently, there exists a region $r \in \mathcal{R} \setminus \{a, b\}$ such that $p_e \in r$. Hence, $(p_e, r) \in X_{E, \mathcal{R}}$ and thus $|X_{E, \mathcal{R}}| \geq 1$. Assume that the claim holds for $|E| - \frac{dn^{1+\alpha}}{2} = i$, where $i > 1$, and consider the case that $|E| - \frac{dn^{1+\alpha}}{2} = i + 1$. Let $e = (a, b) \in E$ be an edge such that there exists a region $r \in \mathcal{R} \setminus \{a, b\}$ with $p_e \in r$. Define $E' = E \setminus \{e\}$. Note that $P_{E'} \subset P_E$ and $|E'| - \frac{dn^{1+\alpha}}{2} = i$. By the induction hypothesis it follows that $|X_{E', \mathcal{R}}| \geq |E'| - \frac{dn^{1+\alpha}}{2}$. Observe that $X_{E', \mathcal{R}} \subset X_{E, \mathcal{R}}$ and that $|X_{E, \mathcal{R}}| \geq |X_{E', \mathcal{R}}| + 1$. It follows that

$$|X_{E, \mathcal{R}}| \geq |X_{E', \mathcal{R}}| + 1 \geq |E'| - \frac{dn^{1+\alpha}}{2} + 1 = i + 1 = |E| - \frac{dn^{1+\alpha}}{2}.$$

\blacksquare

Observation 8 Let $0 \leq \alpha \leq 1$ and let X be a binomial random variable with parameters n and p . Then

$$\mathbf{E}[X^{1+\alpha}] \leq \mathbf{E}[Xn^\alpha] = n^\alpha \mathbf{E}[X] = n^{1+\alpha}p.$$

Lemma 5. Let $0 \leq \alpha \leq 1$ and let c and d be the constants of Lemma 3. Let \mathcal{R} be a set of n simple closed Jordan regions such that $|\mathcal{U}_{\mathcal{R}}(m)| \leq cm^{1+\alpha}$ for $1 \leq m \leq n$. Let $E \subseteq E(I(\mathcal{R}))$ such that $|E| > dn^{1+\alpha}$ and let $\{p_e | e \in E\}$ and $X_{E,\mathcal{R}}$ be as before. Then $|X_{E,\mathcal{R}}| \geq \frac{|E|^2}{2dn^{1+\alpha}}$.

Proof: Let $\mathcal{R}' \subseteq \mathcal{R}$ be a subset of regions of \mathcal{R} chosen randomly and independently such that for every region $r \in \mathcal{R}$, $\Pr[r \in \mathcal{R}'] = p$ for $p = \frac{dn^{1+\alpha}}{|E|}$ (note that $p < 1$). Let $E' \subseteq E$ be the subset of edges that is defined by the intersections of regions in \mathcal{R}' . Let $P_{E'} = \{p_e : e \in E'\}$. $P_{E'} \subseteq P_E$ and thus $X_{E',\mathcal{R}'} \subseteq X_{E,\mathcal{R}}$. Each of $|\mathcal{R}'|$, $|E'|$, and $|X_{E',\mathcal{R}}|$ is a random variable.

By Lemma 4 and by linearity of expectation, it follows that $\mathbf{E}[|X_{E',\mathcal{R}'}|] \geq \mathbf{E}[|E'|] - \mathbf{E}[\frac{d}{2}|\mathcal{R}'|^{1+\alpha}]$.

By Observation 8, $\mathbf{E}[\frac{d}{2}|\mathcal{R}'|^{1+\alpha}] \leq \frac{d}{2}n^{1+\alpha}p$. Hence, it follows that $\mathbf{E}[|X_{E',\mathcal{R}'}|] \geq \mathbf{E}[|E'|] - \frac{d}{2}n^{1+\alpha}p$. For an edge $e = (a, b) \in E$, $\Pr[e \in E'] = \Pr[a, b \in \mathcal{R}'] = p^2$ so $\mathbf{E}[|E'|] = p^2|E|$. In addition, for an edge $e = (a, b)$ and a region $r \in \mathcal{R} \setminus \{a, b\}$ $\Pr[(p_e, r) \in X_{E',\mathcal{R}'}] = \Pr[a, b, r \in \mathcal{R}'] = p^3$. Thus, $\mathbf{E}[|X_{E',\mathcal{R}'}|] = p^3|X_{E,\mathcal{R}}|$. It follows that $|X_{E,\mathcal{R}}| \geq \frac{|E|}{p} - \frac{dn^{1+\alpha}}{2p^2}$. Substituting the value of p in the latter inequality completes the proof of the lemma. ■

Next, a proof of Lemma 1 is presented.

Proof of Lemma 1. Let d be the constant from Lemma 3. Let $V \subseteq V(G_k(\mathcal{R}))$ be a subset of m vertices and let G be the subgraph of $G_k(\mathcal{R})$ induced by V . Define $E = E(G)$. Observe that $E \subseteq E(I(\mathcal{R}))$. There are two cases: Either $|E| \leq dm^{1+\alpha}$ or $|E| > dm^{1+\alpha}$. In the former case, the average degree of a vertex in G is at most $2dm^\alpha$. In the latter case, it follows from Lemma 5 that $|X_{E,V}| \geq \frac{|E|^2}{2dm^{1+\alpha}}$. On the other hand, since $E \subseteq E(G_k(\mathcal{R}))$ then by definition, for every edge $e \in E$ the chosen point p_e can belong to at most k other regions of \mathcal{R} . Thus $|X_{E,V}| \leq k|E|$. Combining these two inequalities we have: $|E| \leq 2dkm^{1+\alpha}$ and thus the average degree of G in this case is at most $4dkm^\alpha$. Hence, in G there exists a vertex whose degree is at most $\max\{2dm^\alpha, 4dkm^\alpha\} = 4dkm^\alpha$. ■

As mentioned in the introduction, Theorem 4 is a corollary of a combination of Theorems 1, 2, and Theorem 3.

3.2 k -Strong Conflict-Free Coloring of Axis-Parallel Rectangles

In this section, we consider $kSCF$ -colorings of axis-parallel rectangles and prove Theorems 5 and 6. As mentioned in the introduction, a proof of Theorem 5 can be derived from a combination of Theorem 1 and Theorem 6. Consequently, we concentrate on a proof of Theorem 6. To that end we require the following lemma.

Lemma 6. Let $k \geq 2$. Let \mathcal{R} be a set of n axis-parallel rectangles such that all rectangles in \mathcal{R} intersect a common vertical line ℓ , and let $H = H(\mathcal{R})$. Then $c_H(k) = O(k)$.

Proof: Assume, without loss of generality, that the rectangles are in general position (that is, no three rectangles' boundaries intersect at a common point). According to Theorem 3, it is sufficient to prove that for every subset of rectangles $\mathcal{R}' \subseteq \mathcal{R}$, the union-complexity of \mathcal{R}' is at most $O(|\mathcal{R}'|)$. Let $\mathcal{R}' \subseteq \mathcal{R}$ and consider the boundary of the union of the rectangles of \mathcal{R}' that is to the right of the line ℓ . Let $\partial_r \mathcal{R}'$ denote this boundary. An intersection point on $\partial_r \mathcal{R}'$ results from the intersection of a horizontal side of a rectangle and a vertical side of another rectangle. Each horizontal side of a rectangle in \mathcal{R}' may contribute at most one intersection point to $\partial_r \mathcal{R}'$. Indeed, let s be a horizontal rectangle side. Let p be the right-most intersection point on s to the right of the line ℓ and let q be any other intersection point on s to the right of ℓ . Let r be the rectangle whose vertical side defines p on s . Since r intersects ℓ , the point p lies on the right vertical side of r . Hence, $q \in r$; for otherwise either r does not intersect ℓ or q is to the left of ℓ , in which case q does

not lie on $\partial_r \mathcal{R}'$. It follows that every horizontal side s of some rectangle contributes at most one point to $\partial_r \mathcal{R}'$. As there are $2|\mathcal{R}'|$ such sides then $\partial_r \mathcal{R}'$ contains at most $2|\mathcal{R}'|$ points. A symmetric argument holds for the boundary of $\partial \mathcal{R}'$ that lies to the left of ℓ . Hence, the union-complexity of \mathcal{R}' is at most $O(|\mathcal{R}'|)$. By Theorem 3, the claim follows. ■

Next, we prove Theorem 6.

Proof of Theorem 6. Let ℓ be a vertical line such that at most $n/2$ rectangles lie fully to its right and to its left, respectively. Let \mathcal{R}' and \mathcal{R}'' be the sets of rectangles that lie entirely to the right and entirely to the left of ℓ , respectively. Let \mathcal{R}_ℓ denote the set of rectangles in \mathcal{R} that intersect ℓ , and let $c(n)$ denote the least number of colors required by a colorful coloring of any n axis-parallel rectangles. By Lemma 6, the set of rectangles \mathcal{R}_ℓ can be colored using $O(k)$ colors. In order to obtain a k -colorful coloring of \mathcal{R} , we color \mathcal{R}_ℓ using a set D of $O(k)$ colors. We then color \mathcal{R}' and \mathcal{R}'' recursively by using the same set of colors D' such that $D \cap D' = \emptyset$. The function $c(n)$ satisfies the recurrence $c(n) \leq O(k) + c(n/2)$. Thus, $c(n) = O(k \log n)$. Let φ be the resulting coloring of the above coloring procedure. It remains to prove that φ is a valid k -colorful coloring of \mathcal{R} . The proof is by induction on the cardinality of \mathcal{R} . Suppose \mathcal{R}' and \mathcal{R}'' are colored correctly under φ , and consider a point $p \in \bigcup_{r \in \mathcal{R}} r$. If $r(p) \subset \mathcal{R}_\ell$ or $r(p) \subset \mathcal{R}'$ or $r(p) \subset \mathcal{R}''$ then by Lemma 6 and the induction hypothesis, $r(p)$ is colored correctly under φ . It is not possible that both $r(p) \cap \mathcal{R}' \neq \emptyset$ and $r(p) \cap \mathcal{R}'' \neq \emptyset$. Hence, it remains to consider points p for which either $r(p) \subset \mathcal{R}_\ell \cup \mathcal{R}'$ or $r(p) \subset \mathcal{R}_\ell \cup \mathcal{R}''$. Consider a point p which is, w.l.o.g, of the former type. Let $i = |r(p) \cap \mathcal{R}_\ell|$ and $j = |r(p) \cap \mathcal{R}'|$. If either $i \geq k$ or $j \geq k$, then either by Lemma 6 or by the inductive hypothesis the hyperedge $r(p)$ is k -colorful. It remains to consider the case that $i + j \geq k$ and $i, j < k$. Let φ_ℓ and $\varphi_{\mathcal{R}'}$ be the colorings of \mathcal{R}_ℓ and \mathcal{R}' induced by φ , respectively. By the inductive hypothesis, the rectangles in the set $r(p) \cap \mathcal{R}'$ are colored distinctively using j colors under $\varphi_{\mathcal{R}'}$. In addition, by Lemma 6, the rectangles in $r(p) \cap \mathcal{R}_\ell$ are colored using i distinct colors under φ_ℓ . Moreover, the colors used in φ_ℓ are distinct from the ones used in $\varphi_{\mathcal{R}'}$. Hence, $r(p)$ is $\min\{|r(p)|, k\}$ -colorful. This completes the proof of the lemma. ■

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